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MODELS OF BIAS OF MEAN SQUARE VALUE DIGITAL ESTIMATOR FOR SELECTED DETERMINISTIC AND RANDOM SIGNALS

The article presents the probability density functions as well as characteristic functions of selected periodic and random signals. On their basis, original models of the bias of the mean square value digital estimator have been designed. These models were employed in investigating estimation errors caused by analog to digital conversion and analog to digital conversion with a dither signal. Selected graphical model representations and their analyses are demonstrated.

It has been shown that for a triangular probability density function random signal with the amplitude $A_t = kq$, $k \in N \setminus \{0\}$, mean square value reconstruction occurs on the basis of a signal quantized with the accuracy of Sheppard's correction. Whereas for periodic signals as well as for the sum of periodic and random signals, the δ component of bias due to the nonsatisfaction of the reconstruction condition is a suppressed oscillating function of the quotient of the amplitude A and the quantization step size q.

It has been proved that by adding, prior to quantization, a triangular distribution random signal with zero mean and the amplitude $A_t = kq$ ($k = 1, 2, \ldots$) in the mean square value measurement of any periodic signal, this bias component can be brought to zero.

Keywords: probability density function, characteristic function, Widrow's theory of quantization, mean square value, estimator bias

1. INTRODUCTION

In most contemporary devices and measurement systems, the investigated signals are the subject of digitization in the time domain (sampling) and in the value domain (quantization). The distortion accompanying quantization may decide the measurement accuracy.

Modeling parameter estimation errors and signal characteristics caused by quantization requires familiarity with characteristic functions of the investigated signals. These functions can be obtained on the basis of their probability density. In the literature, one can find expressions describing characteristic functions as well as errors corresponding to only few types of signals, e.g. rectangular and normal distribution signals, a sinusoidal signal, as well as a combination of sinusoidal and Gaussian signals [1-6].

Considered in the article are rectangular, normal as well as triangular distribution random signals; periodic signals: sinusoidal, rectangular, triangular and sawtooth as well as sums of random and periodic signals. Sinusoidal, rectangular, triangular and sawtooth signals are basic periodic signals whereas rectangular, normal and triangular pdf random signals, after addition to periodic signals, can function as dithers.

Presented are the quantization theorem and the consequences due to it not being satisfied in mean square value measurement.

Chapter 4 is the most important part of the work. It contains mathematical models of the bias of the mean square value estimators of the above mentioned signals, their graphical representations and analyses. Calculations of the bias were carried out in accordance with the IEEE 754 standard and did not influence considerably the research accuracy [7].

2. PROBABILITY DENSITY FUNCTIONS AND CHARACTERISTIC FUNCTIONS FOR SELECTED SIGNALS

The probability density function (PDF) determination can be facilitated by using following theorem:

Theorem I. If there are given the random variable T with the probability density function $f(t)$, and the multivalued function $x = h(t)$, then the new random variable X has the density [8]:

$$p(x) = f(h^{-1}(x))|h^{-1}(x)'|, \tag{1}$$

where $h^{-1}(x)$ is an inverse function of $h(t)$.

Except for a rectangular signal whose envelope is not a multivalued function, density distribution functions of periodic signals can be calculated on the basis of Theorem I. In [7], it has been shown that for such signals:

$$f(h^{-1}(x)) = \frac{\omega}{\pi}. \tag{2}$$

Let us consider, by way of example, a triangular signal with the amplitude A and the duty factor $\lambda = 0.5$. The inverse function of the envelope of this signal can be expressed by the formula:

$$t = h^{-1}(x) = \frac{\pi}{2\mathrm{A}\omega} x. \tag{3}$$

Moreover:

$$h^{-1}(x)' = \frac{\pi}{2\mathrm{A}\omega}. \tag{4}$$

Utilizing formulae (1 – 4), the PDF of a triangular signal can be expressed in the form:

$$p(x)=\frac{1}{2A}. \tag{5}$$

The same PDF can be ascribed to a sawtooth signal. The properties of the triangular signal quantization errors presented in the article are also valid for a sawtooth signal.

The characteristic function (CF) is the Fourier transform of the probability density function *p*(*x*) into the *v* domain with a sign inversion. For the one-dimensional continuous random variable *X*, it equals:

$$\Phi_x(v)=\int_{-\infty}^{\infty} p(x)e^{jvx}dx, \tag{6}$$

whereas for the jump random variable *X*, it is expressed by formula [5]:

$$\Phi_\xi(v)=\sum_{x\in S_\xi} P(\xi=x)e^{jvx}, \tag{7}$$

where S_ξ is a set of distribution jump points.

Collected in Table 1 are relations for density and characteristic functions of the signals being investigated [2, 3, 9], with 1 corresponding to the density and characteristic functions of a uniform distribution random signal with the amplitude A_u; 2 – to a random Gaussian distribution signal with the standard deviation σ_n; 3 – to a random triangular distribution signal with the amplitude A_t; 4 – to a sinusoidal signal with the amplitude *A*; 5 – to a rectangular signal with the amplitude *A* and the duty factor λ; 6, 7 – to a triangular or a sawtooth signal with the amplitude *A*. All the signals have zero mean values.

Table 1. Probability density functions (PDF's) and characteristic functions (CF's).

No.	PDF	CF
1	$p(x)=\begin{cases}1/2A_u, & \lvert x\rvert\le A_u\\ 0, & elsewhere\end{cases}$	$\Phi_x(v)=\frac{\sin(A_u v)}{A_u v}$
2	$p(x)=\frac{1}{\sigma_n\sqrt{2\pi}}\exp\left(-0.5\left(\frac{x}{\sigma_n}\right)^2\right)$	$\Phi_x(v)=\exp(-0.5v^2\sigma_n^2)$
3	$p(x)=\begin{cases}\frac{1}{A_t}\left(1-\frac{\lvert x\rvert}{A_t}\right), & \lvert x\rvert\le A_t\\ 0, & elsewhere\end{cases}$	$\Phi_x(v)=\frac{2-2\cos(A_t v)}{A_t^2 v^2}$
4	$p(x)=\frac{1}{\pi\sqrt{A^2-x^2}}$	$\Phi_x(v)=J_0(Av)$
5	$p(x)=\begin{cases}1-\lambda, & x=-A\\ \lambda, & x=A\\ 0, & elsewhere\end{cases}$	$\Phi_x(v)=(1-\lambda)e^{-jvA}$ $+\lambda e^{jvA}=\cos(Av)$ $+j\lvert(2\lambda-1)\sin(Av)\rvert;\ j=\sqrt{-1}$
6 7	$p(x)=\begin{cases}1/2A, & \lvert x\rvert\le A\\ 0, & elsewhere\end{cases}$	$\Phi_x(v)=\frac{\sin(Av)}{Av}$

The formulas included in the Table 1 are applied in Chapter 4 to determine the bias of mean square value estimator.

3. MEAN SQUARE VALUE RECONSTRUCTION CONDITIONS

One of the two Widrow's quantization theorems referring to reconstruction of moments can be expressed as follows:

Theorem II (Widrow's). If the CF of x is band-limited so that [10]

$$\Phi_x(v)=0,\ |v|\geq\frac{2\pi}{q}-\varepsilon, \tag{8}$$

with ε positive and arbitrarily small, then the moments of x can be calculated (reconstructed) from the moments of x_q, where x and x_q are signals before and after quantization, and q is the quantization step size.

The mean square value can be derived by differentiating the characteristic function:

$$E[x^2]=-\frac{d^2\Phi_x(v)}{dx^2}\Big|_{v=0}. \tag{9}$$

If the quantization characteristic is of the roundoff type, then the characteristic functions $\Phi_{xq}(v)$, corresponding to the signal which has been quantized, can be derived from formulae [10, 11]:

$$\Phi_{xq}(v)=\sum_{i=-\infty}^{\infty}\Phi_x\left(v-\tfrac{2\pi}{q}i\right)\quad \sin c\left[\tfrac{q}{2}\left(v-\tfrac{2\pi}{q}i\right)\right]. \tag{10}$$

The theorem assumption being satisfied, the following relation is true [10, 11]:

$$E[x_q^2]=E[x^2]+\frac{q^2}{12}, \tag{11}$$

linking the signal mean square value $E[x^2]$ with its estimator $E[x_q^2]$, obtained on the basis of a quantized signal.

Whereas the calculated on the basis of (9) and (10) the mean square value of the quantized signal x_q assumes the form [4]:

$$E[x_q^2]=E[x^2]+\frac{q^2}{12}+\frac{q}{\pi}\sum_{\substack{i=-\infty\\ i\neq 0}}^{\infty}\dot{\Phi}_x\left(\frac{2\pi}{q}i\right)\frac{(-1)^{i+1}}{i}+\frac{q^2}{2\pi^2}\sum_{\substack{i=-\infty\\ i\neq 0}}^{\infty}\Phi_x\left(\frac{2\pi}{q}i\right)\frac{(-1)^i}{i^2}. \tag{12}$$

If condition (8) occurred, then relation (12) would assume the form of (11), which means that the bias component:

$$b = E[x_q^2] - E[x^2] - \frac{q^2}{12} = \frac{q}{\pi} \sum_{\substack{i=-\infty \\ i \neq 0}}^{\infty} \dot{\Phi}_x\left(\frac{2\pi}{q} i\right) \frac{(-1)^{i+1}}{i} + \frac{q^2}{2\pi^2} \sum_{\substack{i=-\infty \\ i \neq 0}}^{\infty} \Phi_x\left(\frac{2\pi}{q} i\right) \frac{(-1)^i}{i^2}. \tag{13}$$

would assume a zero value.

The component b also assumes a zero value when:

$$\Phi_x\left(\frac{2\pi}{q} i\right) = 0 \quad \text{and} \quad \dot{\Phi}_x\left(\frac{2\pi}{q} i\right) = 0 \quad \text{for } i \in C \setminus \{0\} \tag{14}$$

Expressed in relation to $E[x^2]$, it is of the form:

$$\delta = \frac{b}{E[x^2]}. \tag{15}$$

The relative component δ_{sh} due to ignoring Sheppard's q^2 / 12 correction in the result can be expressed by the relation:

$$\delta_{sh} = \frac{q^2}{12\, E[x^2]}. \tag{16}$$

The relations (15) and (16) are applied in Chapter 4.

4. INFLUENCE OF QUANTIZATION ON THE BIAS OF MEAN SQUARE VALUE ESTIMATOR

4.1. Random signals

Using Table 1 and relations (13) and (15), for uniform, Gaussian and triangular distribution random signals, the mean square value bias components δ can be calculated. They are of the form:

$$\delta_1(\alpha) = 3\sum_{i=1}^{\infty} (-1)^i \left[\left(\frac{1}{\pi i}\alpha^{-1}\right)^3 \sin(2\pi i\alpha) - \left(\frac{1}{\pi i}\alpha^{-1}\right)^2 \cos(2\pi i\alpha) \right], \tag{17}$$

$$\delta_2(\beta) = 2\sum_{i=1}^{\infty} (-1)^i \exp\{-2(\pi i\beta)^2\} \left[2 + \frac{1}{2}\left(\frac{1}{\pi i}\beta^{-1}\right)^2 \right], \tag{18}$$

$$\delta_3(\chi) = 6\sum_{i=1}^{\infty} (-1)^i \left(\frac{1}{\pi i}\right)^3 \chi^{-3} \left[\frac{3}{2\pi i}\chi^{-1}(1 - \cos(2\pi i\chi)) - \sin(2\pi i\chi) \right], \tag{19}$$

where $\alpha = A_u/q$, $\beta = \sigma_n/q$, $\chi = A_t/q$.

Shown in Fig. 1 are diagrams of the components δ and δ_{sh} of the mean square value estimator bias of uniform, Gaussian and triangular distribution random signals due to the nonsatisfaction of the reconstruction condition as well as to the nonapplication of Sheppard's correction. The error subscripts correspond to the signal numbers in Table 1.

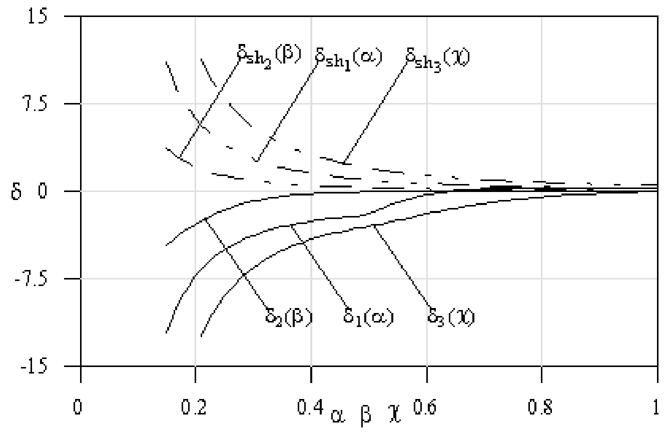

Fig. 1. Normalized biases of the mean square value estimator of an uniformly distributed signal (1), Gaussian signal (2) and triangular PDF (3) signal.

For a rectangular PDF random signal with the amplitude $A_u = 0.5kq$, $k \in N \setminus \{0\}$, $\Phi_x\left(\frac{2\pi}{q} i\right) = 0$, $\dot{\Phi}_x\left(\frac{2\pi}{q} i\right) \neq 0$ occur, and the error δ_1 equals:

$$\delta_1(0.5)=3\left(\frac{2}{\pi}\right)^2 \sum_{i=1}^{\infty} (-1)^{i+1} \frac{\cos(\pi i)}{i^2} = -2. \tag{20}$$

In the case of a normal distribution random signal, one cannot select such a σ_n for which $\Phi_x\left(\frac{2\pi}{q} i\right) = 0$ and $\dot{\Phi}_x\left(\frac{2\pi}{q} i\right) = 0$, $i \in \mathbf{C} \setminus \{0\}$. It should be noted, however, that the CF approaches zero fast and for $\sigma_n = q$ the error can be estimated as follows [2, 3]:

$$\delta_2(1) \approx 1.1 \cdot 10^{-8}. \tag{21}$$

For a triangular PDF signal with the amplitude $A_t = kq$, $k \in N \setminus \{0\}$, the relations:

$$\Phi_x\left(\frac{2\pi}{q} i\right)=0 \ , \ \dot{\Phi}_x\left(\frac{2\pi}{q} i\right)=\frac{2\sin\left(A_t \frac{2\pi}{q} i\right)}{A_t\left(\frac{2\pi}{q} i\right)^2}-\frac{4-4\cos\left(A_t \frac{2\pi}{q} i\right)}{A_t^2\left(\frac{2\pi}{q} i\right)^3}=0, \tag{22}$$

are true and:

$$\delta_3(k)=0, k \in N\backslash(0), \tag{23}$$

which signifies that mean square value reconstruction occurs on the basis of a signal quantized with the accuracy of Sheppard's $q^2 / 12$ correction, i.e. according to formula (11).

4.2. Periodic signals

Using Table 1 as well as relations (13) and (15), the mean square value bias relative components δ of the periodic signals: sinusoidal with the amplitude A, rectangular with the amplitude A and the signal duty λ, as well as triangular (sawtooth) with the amplitude A, can be expressed by the relations:

$$\delta_4(\gamma) = 4\sum_{i=1}^{\infty}(-1)^i\left[\frac{1}{\pi i}\gamma^{-1}J_1(2\pi i\gamma)+\frac{1}{2}\left(\frac{1}{\pi i}\gamma^{-1}\right)^2 J_0(2\pi i\gamma)\right], \tag{24}$$

$$\begin{aligned}\delta_5(\gamma)=&2\sum_{i=1}^{\infty}(-1)^i\left[\frac{1}{2}\left(\frac{1}{\pi i}\gamma^{-1}\right)^2\cos(2\pi i\gamma)+\frac{1}{\pi i}\gamma^{-1}\sin(2\pi i\gamma)\right]+\\&+j(2\lambda-1)2\sum_{i=1}^{\infty}(-1)^i\left[\frac{1}{2}\left(\frac{1}{\pi i}\gamma^{-1}\right)^2\sin(2\pi i\gamma)-\frac{1}{\pi i}\gamma^{-1}\cos(2\pi i\gamma)\right],\end{aligned} \tag{25}$$

$$\delta_{6,7}(\gamma)=3\sum_{i=1}^{\infty}(-1)^i\left[\left(\frac{1}{\pi i}\gamma^{-1}\right)^3\sin(2\pi i\gamma)-\left(\frac{1}{\pi i}\gamma^{-1}\right)^2\cos(2\pi i\gamma)\right], \tag{26}$$

where $\gamma = A/q$, J_0 and J_1 are respectively zero- and first-order Bessel functions of the first kind, $j=\sqrt{-1}$.

Presented in Fig. 2 are courses of the bias relative components δ and δ_{sh} caused by the nonsatisfaction of the reconstruction condition as well as by the nonapplication of Sheppard's correction. The error subscripts correspond to the signal numbers in Table 1. For a rectangular signal, $\lambda = 0.5$ is assumed.

As follows from Fig. 2, for each of the periodic signals being considered, the bias component δ can assume values significantly exceeding – as regards the absolute value – the component δ_{sh} due to the nonapplication of Sheppard's correction.

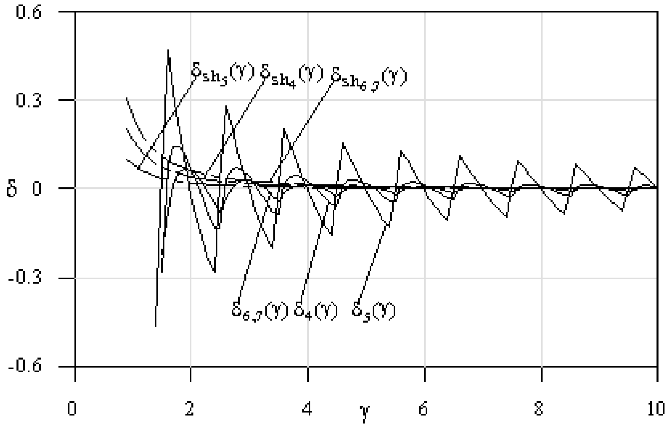

Fig. 2. Normalized biases of the mean square value estimator for sinusoidal (4), square (5), triangular (and sawtooth) (6, 7) signal.

4.3. Sum of periodic and random signals

Random variables corresponding to periodic and random signals are statistically independent, therefore the characteristic function of their sum is the product of the component characteristic functions. Using this property, the CF's in Table 1 and relations (13), (15), models of relative mean square value bias of the sum of periodic signals and uniform, Gaussian and triangular distribution random signals have been derived.

These models can be utilized, for example, in investigating estimation errors caused by analog to digital conversion with a dither signal.

In the case of a sinusoidal signal with an uniformly distributed signal, with a Gaussian signal and with a triangular PDF random signal, we obtain respectively[1]:

$$\delta_8(\gamma)=2\sum_{i=1}^{\infty}(-1)^i\gamma^{-2}\alpha^{-1}J_0(2\pi i\gamma)\sin(2\pi i\alpha)\left(\frac{1}{\pi i}\right)^2\left[\frac{1}{\pi i}-\alpha\, ctg(2\pi i\alpha)+\gamma\frac{J_1(2\pi i\gamma)}{J_0(2\pi i\gamma)}\right], \quad (27)$$

$$\delta_9(\gamma)=2\sum_{i=1}^{\infty}(-1)^i\left\{\left[(2\beta\,\gamma^{-1})^2+\left(\frac{1}{\pi i}\gamma^{-1}\right)^2\right]J_0(2\pi i\gamma)+\frac{2}{\pi i}\gamma^{-1}J_1(2\pi i\gamma)\right\}\exp\left[-2(\pi i\beta)^2\right], \quad (28)$$

$$\delta_{10}(\gamma)=2\sum_{i=1}^{\infty}(-1)^i\chi^{-2}\gamma^{-1}J_0(2\pi i\gamma)\left(\frac{1}{\pi i}\right)^3$$
$$\left[(1-\cos(2\pi i\chi))\left(\frac{J_1(2\pi i\gamma)}{J_0(2\pi i\gamma)}+\frac{3}{2\pi i}\gamma^{-1}\right)-\chi\,\gamma^{-1}\sin(2\pi i\chi)\right]. \quad (29)$$

[1] The errors have been normalized to the deterministic signal mean square value.

For comparison of the influence of random signals on the estimation accuracy of the periodic signal mean square value, the equality of the variances of these signals has been assumed [13]:

$$\sigma_n^2 = \frac{A_u^2}{3} = \frac{A_t^2}{6}. \tag{30}$$

Moreover, $\sigma_n = \sigma_u = \sigma_t = 0.3q$ as well as $\sigma_n = \sigma_u = \sigma_t = 0.5q$ have been successively assumed, therefore the amplitudes of the uniform and the triangular distribution random signals have been selected on the basis of the relations $A_u = 0.3\sqrt{3}q$ and $A_t = 0.3\sqrt{6}q$ as well as $A_u = 0.5\sqrt{3}q$ and $A_t = 0.5\sqrt{6}q$, respectively.

In Figure 3 courses of the bias components δ are presented.

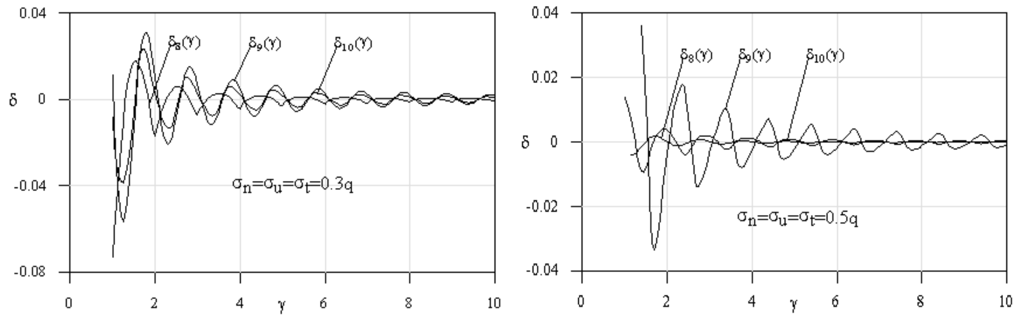

Fig. 3. Normalized biases of the mean square value estimator for sinusoidal signal with uniformly distributed signal (8), Gaussian signal (9) and triangular PDF signal (10).

For a rectangular signal with an uniformly distributed signal, with a Gaussian signal and with a triangular PDF random signal, we obtain:

$$\begin{aligned}\delta_{11}(\gamma) = &\sum_{i=1}^{\infty}(-1)^i \gamma^{-2}\cos(2\pi i\gamma)\sin(2\pi i\alpha)\left(\frac{1}{\pi i}\right)^2 \\ &\left[\frac{1}{\pi i}\alpha^{-1} + \gamma\alpha^{-1}tg(2\pi i\gamma) - ctg(2\pi i\alpha)\right] + \\ &+ j(2\lambda - 1)\sum_{i=1}^{\infty}(-1)^i \gamma^{-2}\sin(2\pi i\gamma)\sin(2\pi i\alpha)\left(\frac{1}{\pi i}\right)^2 \\ &\left[\frac{1}{\pi i}\alpha^{-1} - \gamma\alpha^{-1}ctg(2\pi i\gamma) - ctg(2\pi i\alpha)\right],\end{aligned} \tag{31}$$

$$\delta_{12}(\gamma) = 2\sum_{i=1}^{\infty}(-1)^i \gamma^{-2}\cos(2\pi i\gamma)\exp\left\{-2(\pi i\beta)^2\right\} \left[2\beta^2 + \frac{1}{\pi i}\gamma tg(2\pi i\gamma) + \frac{1}{2}\left(\frac{1}{\pi i}\right)^2\right] + + j(2\lambda-1)2\sum_{i=1}^{\infty}(-1)^i \gamma^{-2}\sin(2\pi i\gamma)\exp\left\{-2(\pi i\beta)^2\right\} \left[2\beta^2 - \frac{1}{\pi i}\gamma ctg(2\pi i\gamma) + \frac{1}{2}\left(\frac{1}{\pi i}\right)^2\right], \tag{32}$$

$$\delta_{13}(\gamma) = \sum_{i=1}^{\infty}(-1)^i \gamma^{-1}\chi^{-2}\cos(2\pi i\gamma)\left(\frac{1}{\pi i}\right)^3 \left\{(1-\cos(2\pi i\chi))\left[\frac{3}{2\pi i}\gamma^{-1} + tg(2\pi i\gamma)\right] - \sin(2\pi i\chi)\right\} + + j(2\lambda-1)\sum_{i=1}^{\infty}(-1)^i \gamma^{-1}\chi^{-2}\sin(2\pi i\gamma)\left(\frac{1}{\pi i}\right)^3 \left\{(1-\cos(2\pi i\chi))\left[\frac{3}{2\pi i}\gamma^{-1} - ctg(2\pi i\gamma)\right] - \sin(2\pi i\chi)\right\}, \tag{33}$$

Presented in Fig. 4 are courses of the relative component δ for $\lambda = 0.5$ corresponding to Eqs. (31) – (33).

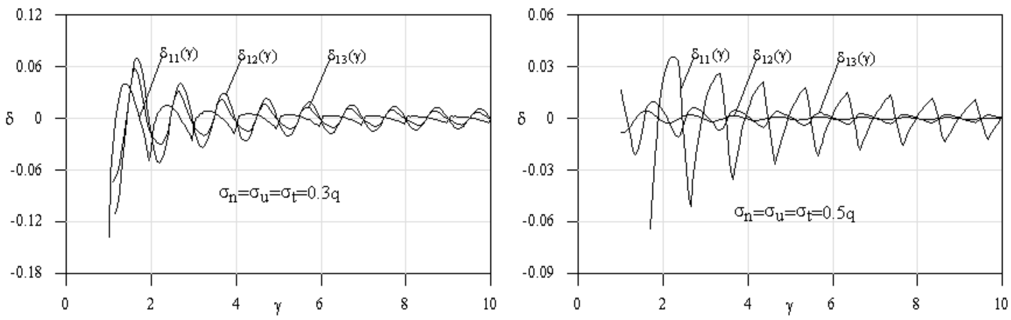

Fig. 4. Normalized biases of the mean square value estimator for a square signal with uniformly distributed signal (11), Gaussian signal (12) and triangular PDF signal (13).

In the case of a triangular (sawtooth) signal with an uniformly distributed signal, with a Gaussian signal and with a triangular PDF random signal, we obtain:

$$\delta_{14}(\gamma)=\frac{3}{2}\sum_{i=1}^{\infty}(-1)^{i}\gamma^{-3}\alpha^{-1}\sin(2\pi i\gamma)\sin(2\pi i\alpha)\left(\frac{1}{\pi i}\right)^{3}\left[\frac{3}{2\pi i}-ctg(2\pi i\gamma)(\gamma+\alpha)\right], \tag{34}$$

$$\delta_{15}(\gamma)=3\sum_{i=1}^{\infty}(-1)^{i}\gamma^{-3}\sin(2\pi i\gamma)\exp\{-2(\pi i\beta)^{2}\}\frac{1}{\pi i}\left[2\beta^{2}+\frac{1}{\pi i}\left(\frac{1}{\pi i}-\gamma\, ctg(2\pi i\gamma)\right)\right], \tag{35}$$

$$\delta_{16}(\gamma)=3\sum_{i=1}^{\infty}(-1)^{i}\gamma^{-2}\chi^{-2}\sin(2\pi i\gamma)\left(\frac{1}{\pi i}\right)^{4}$$
$$\left\{(\cos(2\pi i\chi)-1)\left[\frac{1}{2}ctg(2\pi i\gamma)-\gamma^{-1}\frac{1}{\pi i}\right]-\frac{1}{2}\gamma^{-1}\chi\sin(2\pi i\chi)\right\}. \tag{36}$$

Presented in Fig. 5 are courses of the bias relative component δ corresponding to Eqs. (34)–(36).

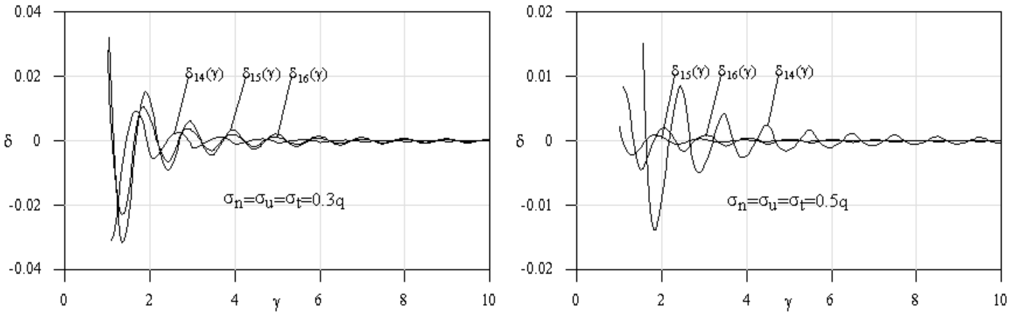

Fig. 5. Normalized biases of the mean square value estimator for triangular (or sawtooth) signal with uniformly distributed signal (14), Gaussian signal (15) and triangular PDF signal (16).

From Figures 3–5 it follows that the largest error decrease in each of the cases being considered is caused by the presence of a Gaussian signal, a slightly smaller error – by the presence of a triangular signal, and the smallest error – by the presence of a uniformly distributed signal. Moreover, in each case, along with random signal level increase, a narrowing down of the error range occurs.

4.4. Triangular PDF random signal as perfect non-subtractive dither

From formulae (22) and (23) it follows that for a triangular distribution random signal x, with zero mean and the amplitude $A_t = kq$, ($k = 1, 2, 3, \ldots$), the relation:

$$\Phi_x\left(\frac{2\pi}{q}i\right)=\dot{\Phi}_x\left(\frac{2\pi}{q}i\right)=0,\ i\in \mathbf{C}\setminus\{0\}, \tag{37}$$

is true.

If prior to quantization a like signal is added to any periodic signal y, then, due to their statistical independence

$$\Phi_{x+y}\left(\frac{2\pi}{q}i\right) = \Phi_x\left(\frac{2\pi}{q}i\right)\Phi_y\left(\frac{2\pi}{q}i\right) = 0, \tag{38}$$

occurs.

Moreover:

$$\dot{\Phi}_{x+y}\left(\frac{2\pi}{q}i\right) = \dot{\Phi}_x\left(\frac{2\pi}{q}i\right)\Phi_y\left(\frac{2\pi}{q}i\right) + \Phi_x\left(\frac{2\pi}{q}i\right)\dot{\Phi}_y\left(\frac{2\pi}{q}i\right) = 0, \tag{39}$$

and according to (13)

$$b = \frac{q}{\pi}\sum_{\substack{i=-\infty \\ i\neq 0}}^{\infty}\dot{\Phi}_{x+y}\left(\frac{2\pi}{q}i\right)\frac{(-1)^{i+1}}{i} + \frac{q^2}{2\pi^2}\sum_{\substack{i=-\infty \\ i\neq 0}}^{\infty}\Phi_{x+y}\left(\frac{2\pi}{q}i\right)\frac{(-1)^i}{i^2} = 0. \tag{40}$$

This means that by adding, prior to quantization, a triangular distribution random signal with zero mean and the amplitude $A_t = kq, (k = 1, 2, 3, \ldots)$ in the mean square value measurement of any periodic signal, the bias component due to the nonsatisfaction of the quantization reconstruction condition can be brought to zero.

5. CONCLUSION

In the article, probability density functions have been presented together with the characteristic functions of selected periodic and random signals and their sums. On their basis, models of the bias of the mean square value digital estimator of signals have been designed.

It has been shown, among other things, that for a Gaussian distribution random signal, the $E[x^2]$ value cannot be reconstructed on the basis of $E[x_q^2]$ with an accuracy of Sheppard's correction, whereas in the case of a triangular distribution random signal with an appropriately selected quantization step size, such a possibility does exist.

For each of the periodic signals being considered, the bias component δ can assume values significantly exceeding – as regards the absolute value – the component δ_{sh} due to the nonapplication of Sheppard's correction.

It has been confirmed that for periodic signals as well as for the sum of periodic and random signals, the δ component is a suppressed oscillating function of the quotient of the amplitude A and the quantization step size q.

For the sum of a periodic signal and a random signal with the standard deviation at $0.3q$ and $0.5q$ levels, the largest bias decrease is caused by the presence of a Gaussian signal, slightly smaller – by the presence of a triangular signal, and the smallest – by the presence of a uniform distribution signal. Moreover, in each case, along with the increase in random signal level, a narrowing down of the error range occurs.

It has been proved that by adding, prior to quantization, a triangular distribution random signal with zero mean and the amplitude $A_t = kq$, ($k = 1, 2, 3, \ldots$) in the mean square value measurement of any periodic signal, the bias component due to the nonsatisfaction of the quantization reconstruction condition can be brought to zero.

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